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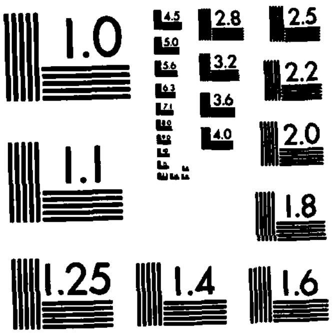

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The interfacial conditions for a cylindrical vortex sheet or a cylindrical fluid layer obtained earlier are examined further for the case of incompressible fluids. Only temporal perturbations are considered. Unlike the single role of destabilization played by the velocity in two-dimensional stratified flows or axisymmetric jet flows, the rotating velocity in vortex motions plays a dual role in flow stability. The radial velocity gradient generates tangential shear and the fluid rotation creates a centrifugal force field. While the former always destabilizes the flow, the latter can either stabilize or destabilize the flow depending upon whether the resultant force field is centrifugally

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20. ABSTRACT (Continued)

stable or unstable. These characteristics are demonstrated by examining three general types of perturbations. We further show that deformations of the vortex sheet or the fluid layer affect the flow field in two ways: disturbing the total pressure field and perturbing the centrifugal force field created by the azimuthal components of the velocity and magnetic fields. The latter, even though it seems to be straightforward, has often been overlooked in previous analyses. It is shown that failure to consider such a perturbation to a stable centrigufal force field will lead to the improper destabilization of modes with smaller axial and azimuthal wave numbers.

CONTENTS

STABILITY CHARACTERISTICS FOR FLOWS OF THE VORTEX SHEET TYPE

INTRODUCTION

The density variations of stratified fluids play a subtle but crucial role in flow characteristics. In stratified shear flows under gravitational influences, for example, the density variations enter the problem through their interaction with the vertical displacements in the flows. The density variation, usually very small in atmospheric and oceanographic studies, is the basis for stability conditions like the Richardson criterion. Such a density variation, no matter how small, cannot be ignored because of its subtle interaction with the perturbation in the vertical direction.

In the case of vortex motions, a centrifugal force field which depends on the radial position will be induced by the rotation of fluids. Because of the complication of the geometry and the dependence of the centrifugal force field on both rotation and density, the "buoyancy" effect and the inertia effect are sometimes difficult to distinguish. The rotation of fluid particles plays a dual role in flow characteristics. While the rotation, interacting with the density, generates a force field to supply radial "buoyancy" effects which can either stabilize or destabilize the flow, the velocity gradient creates shear effects which always destabilize the flow. In other words, two types of instability mechanisms, the centrifugal one and the shear one, are conveyed by the rotation, and they are not as distinct as in the case of two-dimensional parallel flows.

In an early paper on interfacial conditions of a cylindrical vortex sheet, Fung (1980) showed that perturbations to the flow disturb both the pressure field and the centrifugal force field which is created by the fluid rotation and the azimuthal magnetic field. The latter stabilizes or destabilizes the flow depending on whether the force field generated is centrifugally stable or unstable. The perturbation to the centrifugal force field is therefore essential to the flow characteristics especially when rapid changes of flow quantities exist within a thin layer of fluid. Such a perturbation from the centrifugal force field seems to be straightforward but is sometimes easily overlooked. Sufficient care should be taken when analyses involving discontinuous quantities are performed.

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In their analysis of inviscid instability of two-dimensional vortex type flows, Michalke & Timme (1967) failed to consider the effect of perturbations to the centrifugal force field generated by the vortex motion and came to a conclusion all modes were unstable. In a later paper by Leibovich (1969) on stability of inviscid rotating coaxial jets in stratified fluids, the same perturbation to the centrifugal force field was again omitted. The author found that the stabilities uncovered by the analysis were anomalous and therefore concluded that errors might result if a thin but stable fluid layer was replaced by a vortex sheet.

When discontinuities of flow quantities exist in a cylindrical interface, instabilities are likely to occur because of any unbalanced centrifugal forces, and the sharp velocity gradient present at the interface. As mentioned earlier, though the velocity gradient always stabilizes the flow, the centrifugal force induced by vortex motions can either stabilize or destabilized the flows. As for the cases considered by Michalke & Timme (1967) and by Leibovich (1969), the centrifugal force field did have stabilizing effects and should have stabilized modes with smaller wave numbers. This phenomenon will be demonstrated by examining a general class of vortex sheet type flows. The stabilizing or destabilizing effect of the centrifugal force field will be revealed by the perturbation of the field at the interface. A centrifugally stable force field created by the rotation and the azimuthal magnetic field may not always offset the shear instability of the vortex sheet, but certainly will stabilize disturbances corresponding to longer wave lengths.

GOVERNING EQUATIONS AND INTERFACIAL CONDITIONS

The governing stability equations for a general class of incompressible vortex flows with radius-dependent density, velocity and magnetic fields in a cylindrical coordinate system are given as:

$$(N^2 - N_A^2)D^*\left(\frac{u}{N}\right) - \frac{2m}{r}(N\Omega - N_A\Omega_A)\left(\frac{u}{N}\right) = \frac{i}{\rho_o}\left(k^2 + \frac{m^2}{r^2}\right)q \tag{1}$$

$$\left\{(N^2 - N_A^2)[(N^2 - \Phi) - (N_A^2 - \Psi_A)] - 4\Omega^2 N_A^2\left[\left(1 - \frac{\Omega_A}{\Omega}\right)^2 + 2\frac{\Omega_A}{\Omega}\left(1 - \frac{N}{N_A}\right)\right]\right\}\left(\frac{u}{N}\right)$$

$$= \frac{i}{\rho_o}[(N^2 - N_A^2)Dq + \frac{2m}{r}(N\Omega - N_A\Omega_A)q]. \tag{2}$$

Here $N = kW + m\Omega - \omega$ is the Doppler-shifted frequency, $N_A = kW_A + m\Omega_A$ the Alfv́en frequency, ω the complex eigenfrequency, $\rho_o(r)$ the density of the fluid, Ω the angular velocity, W the axial velocity, Ω_A the angular Alfv́en velocity, W_A the axial Alfv́en velocity, u the perturbation velocity in the radial direction, q the perturbation of the total pressure (including the magnetic pressure), k the axial wave number, m the azimuthal wave number, T the surface tension at the interface located at a radial position $r = R$, δ the Dirac Delta function, $D = \frac{d}{dr}$ and $D^* = D + \frac{1}{r}$. The Rayleigh-Synge discriminant is defined as

$$\Phi = \frac{D[\rho_o(r^2\Omega)^2]}{\rho_o r^3}$$

and the Alfv́en discriminant as

$$\Psi_A = \frac{r}{\rho_o} D(\rho_o \Omega^2).$$

The corresponding boundary conditions are the perturbations vanish at the inner and outer boundaries.

The interfacial conditions for possible discontinuities at the vortex sheet or cylindrical fluid layer, obtained by integrating Eqs. (1) and (2) across the interface or by considering the total force balance at the interface, are as follows:

$$< \frac{u}{N} > = 0 \tag{3}$$

$$<q> - i\left\{\frac{u}{N}\right\}_R \left[<\rho_o r(\Omega^2 - \Omega_A^2)> + \frac{T}{R^2}(\kappa^2 + m^2 - 1)\right] = 0 \tag{4}$$

where $\kappa = kR$ and $<\phi> = \phi(R_{+o}) - \phi(R_{-o})$ denotes a possible jump condition at the interface. Readers are referred to Fung (1980) for the detailed derivation and assumption for Eqs. (1) and (2), and the discussion on the two interfacial conditions in Eqs. (3) and (4).

For the convenience of mathematical operations and discussion, we define

$$F_c = <\rho_o r(\Omega^2 - \Omega_A^2)> + \frac{T}{R^2}(\kappa^2 + m^2 - 1) \tag{5}$$

to denote the centrifugal force and surface tension effects at the interface. The interfacial condition (4) now written as

$$<q> - i\left(\frac{u}{N}\right)_R F_c = 0 \tag{6}$$

can then be viewed as the dynamical balance condition between the perturbation of the total pressure (including the magnetic pressure) and the perturbation of the unbalanced centrifugal forces and surface tension at the interface.

A GENERAL TYPE OF VORTEX SHEET

The centrifugal force enters the dynamic interfacial condition through its interaction with the Lagrangian displacement in the r-direction. The interaction represents the influence on flow stability due to the perturbation to the centrifugal force field. To understand such an influence, we will analyze a general class of vortex sheet type flows subject to different types of perturbations. Analytical solutions for some particular flow profiles will be obtained to verify the conclusion of the analysis. Stability domains will also be discussed. The errors that result from omitting the perturbation to the centrifugal force field in earlier analyses [Michalke & Timme (1967); Leibovich (1969)] will be discussed and corrected.

The general type of vortex sheet profile to be considered has two flow regions with their steady state interface located at $r = R$. The flow properties in the inner region are all constant, i.e.,

$$\begin{aligned}
&\rho_o(r) = \rho_1 \\
&\Omega(r) = \Omega_1 \\
&W(r) = W_1 \qquad\qquad \text{for } 0 \leqslant r < R \\
&\Omega_A(r) = \Omega_{A1} \\
&W_A(r) = W_{A1}
\end{aligned}$$

where the quantities with numerical indices are constant. The flow properties in the outer region are arbitrary functions of the radius. The solutions for the perturbation velocity u_1 and the perturbation pressure q_1 in the inner region can be obtained from Eqs. (1) and (2), and the boundary condition at the axis as

$$u_1 = A\,N_1\left\{\frac{2m(N_1\Omega_1 - N_{A1}\Omega_{A1})}{r(N_1^2 - N_{A1}^2)} + \frac{kg_1 I_m'(kg_1 r)}{I_m(kg_1 r)}\right\} I_m(kg_1 r) \tag{7}$$

$$q_1 = -iA(N_1^2 - N_{A1}^2)g_1^2\rho_1 I_m(kg_1 r) \tag{8}$$

where

$$g_1^2 = 1 - 4\left(\frac{N_1\Omega_1 - N_{A1}\Omega_{A1}}{N_1^2 - N_{A1}^2}\right)^2 \tag{9}$$

$$N_1 = kW_1 + m\Omega_1 - \omega$$

$$N_{A1} = kW_{A1} + m\Omega_{A1}$$

and $I_m(kg_1r)$ is the modified Bessel function of the first kind. The prime denotes the total derivative with respect to the argument of the Bessel function. Taking the derivative of Eq. (7) with respect to r, one obtains

$$D^*\left\{\frac{u_1}{N_1}\right\} = A\left\{\frac{2m(N_1\Omega_1 - N_{A1}\Omega_{A1})}{r(N_1^2 - N_{A1}^2)}kg_1I_m'(kg_1r) + \left(\frac{m^2}{r^2} + k^2g_1^2\right)I_m(kg_1r)\right\}. \tag{10}$$

The solutions in the inner region given by Eqs. (7), (8) and (10) will be used to analyze the influence of the centrifugal force field on flow stability subject to three kinds of disturbances at the interface: an axisymmetric perturbation, an aximuthal perturbation, and an arbitrary perturbation.

Case 1: The axisymmetric mode ($m = 0$)

The solutions in the inner region as described by Eqs. (7) and (10) for the axisymmetric case reduce to

$$u_1 = A\ N_1kg_1I_o'(kg_1r) \tag{11}$$

$$D^*u_1 = A\ N_1k^2g_1^2I_o(kg_1r) \tag{12}$$

where

$$N_1 = kW_1 - \omega$$

and

$$g_1 = \left\{1 - 4\left(\frac{N_1\Omega_1 - kW_{A1}\Omega_{A1}}{N^2 - k^2W_{A1}^2}\right)^2\right\}^{1/2}.$$

For the convenience of mathematical operations, we will express the dynamical interfacial condition only in terms of the perturbation velocity. To do this, we substitute (1) into (4) for $m = 0$ and obtain

$$<\rho_o r^2(N^2 - k^2 W_A^2) D^* \left(\frac{u}{N}\right)> + \kappa^2 \left(\frac{u}{N}\right) F_c = 0. \tag{13}$$

The governing stability equation obtained by combining Eqs. (1) and (2) for the axisymmetric mode reduces to the form

$$D\left[\rho_o (N^2 - k^2 W_A^2) D^* \left(\frac{U}{N}\right)\right] - \rho_o k^2 [(N^2 - k^2 W_A^2) g^2 + 4\Omega^2 - \Phi + \Psi_A] \left(\frac{U}{N}\right) = 0 \tag{14}$$

where

$$g^2 = 1 - 4\left(\frac{N\Omega - kW_A \Omega_A}{N^2 - k^2 W_A^2}\right)^2 \quad \text{for } R \leqslant r < \infty.$$

Multiplying Eq. (14) by $r\left(\frac{\bar{u}}{\bar{N}}\right)$, where the quantities with a bar are the complex conjugates, and integrating the resultant equation over the outer region, we obtain, after applying the boundary condition at infinity, the following integral equation

$$\left[r\left(\frac{\bar{u}}{\bar{N}}\right)\rho_o (N^2 - k^2 W_A^2) D^* \left(\frac{u}{N}\right)\right]_{R+o} + \int_R^\infty \rho_o (N^2 - k^2 W_A^2) \left[\left|D^* \left(\frac{u}{N}\right)\right|^2 + k^2 g^2 \left|\frac{u}{N}\right|^2\right] r dr$$

$$- \int_R^\infty \rho_o k^2 (\Phi - 4\Omega^2 - \Psi_A) \left|\frac{u}{N}\right|^2 r dr = 0. \tag{15}$$

Combining Eqs. (11), (12), (13), and (15), we obtain, by using the complex conjugate of Eq. (3),

$$k^2 \kappa I_o'(\kappa \bar{g}_1) \{\rho_1 g_1 I_o(\kappa g_1)(N_1^2 - k^2 W_{A1}^2) - \kappa I_o'(\kappa g_1) F_c\}$$

$$+ \int_R^\infty \rho_o (N^2 - k^2 W_A^2) \{|D^* \phi_k|^2 + k^2 g^2 |\phi_k|^2\} - \int_R^\infty \rho_o k^2 (\Phi - 4\Omega^2 - \Psi_A) |\phi_k|^2 r dr = 0 \tag{16}$$

where the transformation

$$\phi_k = \frac{1}{A\, g_1} \frac{u}{N}$$

has been applied for $R \leqslant r < \infty$.

It is very difficult to observe the general characteristics of Eq. (16) because the arguments of the Modified Bessel functions involve the complex eigenfrequency ω. To observe the effects of the centrifugal force field and other flow quantities on the stability of the flow and especially of the interface, the following special profiles are considered to simplify the integral equation (16). Let

$$W_A = 0 \qquad \text{for } 0 \leqslant r < \infty$$

and

$$\Omega_1 = 0$$

which correspond to a vortex flow field with no axial magnetic flux anywhere and a core which is not rotating. Equation (16) reduces to a quadratic form

$$a_k\left(\frac{\omega}{k}\right)^2 - 2b_k\left(\frac{\omega}{k}\right) + c_k = 0 \tag{17}$$

where

$$a_k = \alpha_0 + \int_R^\infty Q_k dr$$

$$b_k = \alpha_o W_1 + \int_R^\infty W Q_k dr$$

$$c_k = \alpha_o W_1^2 - [\kappa I_o'(\kappa)]^2 F_c + \int W^2 Q_k dr - \int \rho_o(\Phi - \Psi_A)|\phi_k|^2 r dr$$

$$\alpha_o = k^2 \rho_1 \kappa I_o'(\kappa) I_o(\kappa) \geqslant 0$$

and

$$Q_k = \rho_o(|D^*\phi_k|^2 + k^2|\phi_k|^2) r \geqslant 0.$$

Solving for ω, one finds that stability (corresponding to real values of ω) is guaranteed when

$$-\left\{\alpha_k \int_R^\infty (W - W_1)^2 Q_k dr + \delta_k\right\}$$

$$+\left(\alpha_k + \int_R^\infty Q_k dr\right)\left\{\int_R^\infty \rho_o(\Phi - \Psi_A)|\phi_k|^2 r dr + [\kappa I_o'(\kappa)]^2 F_c\right\} \geqslant 0 \tag{18}$$

where

$$\delta_k = \int Q_k dr \int W^2 Q_k dr - \left(\int W Q_k dr\right)^2.$$

Certain characteristics of the flow can be observed from Eq. (18). The first pair of curly brackets contain information on the axial velocities while the second pair of curly brackets contain information on the centrifugal forces at the interface and in the outer region. Since both α_o and Q_k are positive definite, the first term in the first pair of curly brackets represent the axial velocity difference at the interface between the inner and the outer regions, always destabilizing the flow. The second term in the first pair of curly brackets contains information on the axial velocity in the outer region. It can be easily seen from the Schwarz inequality that

$$\delta_k \geqslant 0$$

for all values of W. Therefore we can conclude that the presence of the axial velocity in the outer region always destabilizes the flow except for constant values of axial flows where $\delta_k \equiv 0$. The tangential shears at the interface and in the outer region are suppressed since the perturbations are allowed only in the axial direction. The first term in the second curly bracket is the integral of the Rayleigh-Synge and the Alfv́en discriminants. For vortex flows subject to axisymmetric disturbances and in the absence of the axial velocity and the axial magnetic field, the two discriminants constitute the general Michael condition saying that the necessary and sufficient condition for stability is

$$\Phi - \Psi_A \geqslant 0. \tag{19}$$

Equation (19) represents a state of centrifugal stability [Fung (1980)] and the corresponding integral in the second curly bracket conveys information on centrifugal stability in the outer region. The last term in the curly bracket carries the information on the centrifugal forces acting on both sides of the interface, parallel to the first term in the same brackets. As pointed out by Fung (1980) in his derivation of the dynamic interfacial condition, the jump condition arising from the perturbation of the centrifugal force field is the outcome of integrating the Raleigh-Synge and the Alfvén discriminants across the interface. The last term in the second pair of the curly brackets in Eq. (18) can then be viewed as the integral representation of the generalized Michael condition at the interface. The sign of F_c, indicating whether or not the resultant force at the interface is centrifugally stable, determines the stabilizing or destabilizing effect on the flow. For the present case

$$F_c = R[\rho_2\Omega_2^2 - (\rho_2\Omega_{A2}^2 - \rho_1\Omega_{A1}^2)] + \frac{T}{R^2}(\kappa^2 - 1). \tag{20}$$

The surface tension always stabilizes the flow except for very long axial wave lengths where $\kappa < 1$. It can be shown (Fung 1980) that the centrifugal force balance condition in Eq. (4) can be obtained by integrating Eq. (19) across the interface. Equation (4) can therefore be viewed as the integral representation of the generalized Michael condition, representing a centrifugally stable condition at the interface. As predicted by the generalized Michael condition, the presence of the magnetic field in the inner region stabilizes the flow while that in the outer region destabilizes the flow. The rotational velocity immediately outside the sheet, always stabilizes the flow.

It should be pointed out that the flow profile being considered in this axisymmetric case can be reduced to the one examined by Leibovich (1969) in his study on hydrodynamic stability of inviscid rotating jets, if the axial flow in the outer region and all the magnetic forces in the flow field are deleted. Eq. (18) with $W = \Psi_A = 0$ should have been recovered had the correct interfacial condition described by Eq. (4) been used in his paper. Failure to consider the centrifugal force jump, which contributed considerable stabilizing effects to the flow, led the author to conclude that the flow must be unstable at least to short waves and possibly to all wavelengths. While the vortex sheet type of flows is susceptable to short wave perturbations because of the strong shear effect present at the interface, the centrifugal force jump as in the case investigated by Leibovich (1969) will certainly stabilize those perturbations with longer wavelengths. This characteristic is clearly shown in Eq. (18) and will be supported by an exact solution to be given in the following.

Because of the presence of the centrifugal term involving F_c in Eq. (18), instabilities for large axial wave number can not immediately be concluded. Therefore, it is necessary to obtain solutions for some specific flow profiles in the outer region before the detailed stability phenomena predicted by Eq. (18) can be observed. We will examine the following flow profile

$$\begin{aligned} \rho_o(r) &= \rho_2 \\ W(r) &= W_2 \\ \Omega(r) &= \Omega_2(R/r)^2 \qquad \text{for } R \leqslant r < \infty \\ \Omega_A(r) &= \Omega_{A2} \end{aligned}$$

where ρ_2, W_2, Ω_2, and Ω_{A2} are constants. The perturbation velocity u_2 in the outer region obtained by solving Eq. (14) is

$$u_2 = B(kW_2 - \omega)kK_o'(kr).$$

The stability boundary described by Eq. (18) in this case is

$$- k^2(W_1 - W_2)^2 + \left[\frac{1}{\rho_1} \frac{\kappa I_o'(\kappa)}{I_o(\kappa)} - \frac{1}{\rho_2} \frac{\kappa K_o'(\kappa)}{K_o(\kappa)} \right] F_c \geqslant 0 \tag{21}$$

where F_c is given in Eq. (20). Since the sum of the terms within the square brackets is positive, the stabilizing effect, if any, will come from the centrifugal force term F_c. Even though Eq. (21) is likely to be violated for very large axial wave numbers, perturbations corresponding to smaller axial wave numbers may certainly be stabilized by the centrifugally stable forces at the interface. As demonstrated by Eq. (21), an erroneous conclusion that no axisymmetric modes are stable can easily be reached if the perturbation of the centrifugal force at the interface is omitted as was done by Leibovich (1969). Figure 1 shows the stability domains for the ratio between the centrifugal force and the axial velocity difference as a function of the axial wave number. Even though the flow is unstable for very large axial wave numbers, perturbations corresponding to longer wavelengths will certainly be stabilized by the centrifugal force jump at the interface.

Case 2: The azimuthal mode ($k = 0$)

The solutions in the inner region governed by Eqs. (7) and (10) for azimuthal modes have the forms

$$u_1 = A\, N_1\, r^{m-1} \tag{22}$$

$$D^* u_1 = A\, N_1\, m r^{m-2} \tag{23}$$

where

$$N_1 = m\Omega_1 - \omega.$$

We will again express the dynamic interfacial condition only in terms of the perturbation velocity. Combining Eqs. (1) and (4) for $k = 0$ yields

$$< \rho_o r^2 (N^2 - m^2 \Omega_A^2) D^* \left\{ \frac{u}{N} \right\} > \tag{24}$$

$$+ \left\{ \frac{u}{N} \right\}_R \left\{ - < 2mr\rho_0 (N\Omega - m\Omega_A^2) > + m^2 F_c \right\} = 0.$$

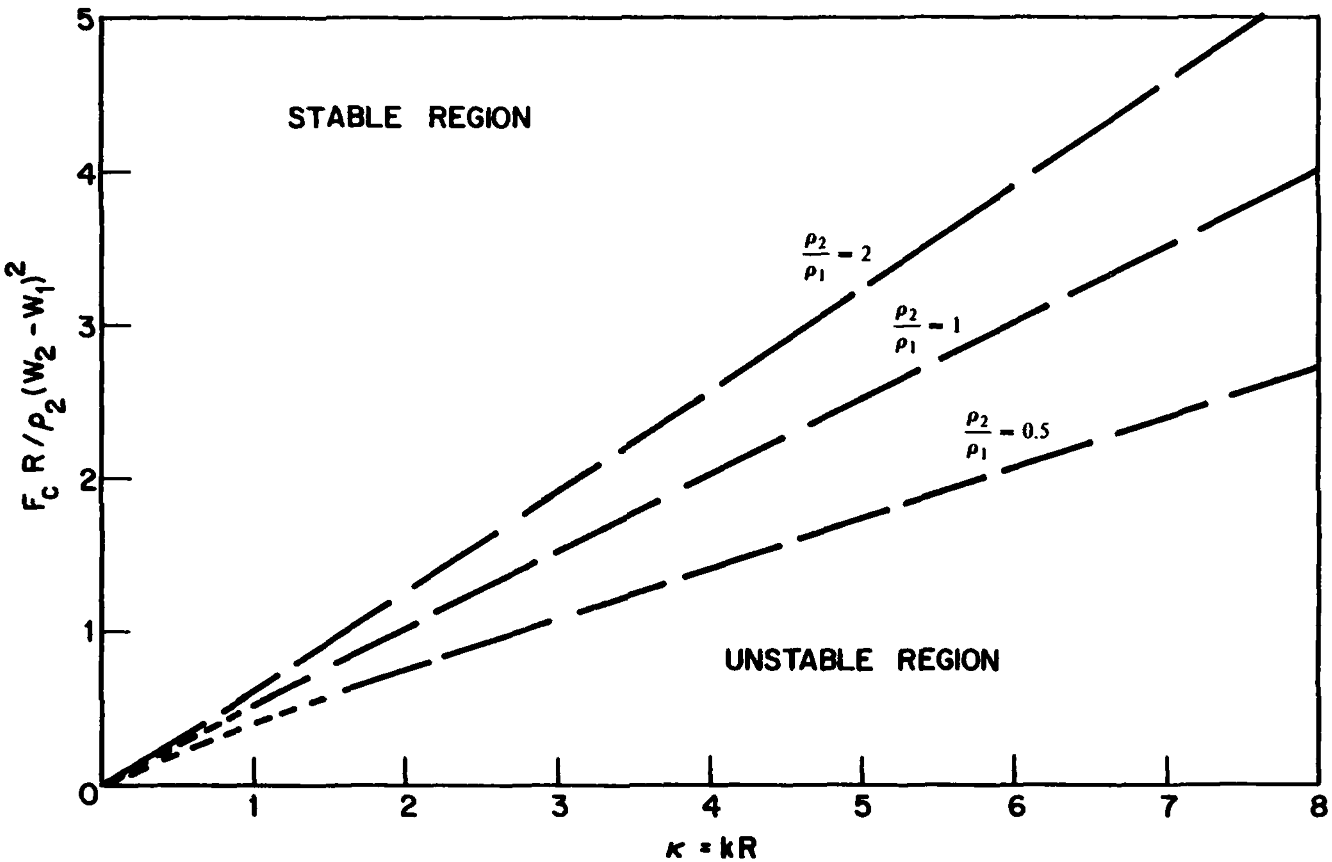

Fig. 1. Stability domains for vortex flows subject to axisymmetric perturbations

As for the outer region, the governing equation obtained from Eqs. (1) and (2) for azimuthal modes is

$$D\left[\rho_0 r^2 (N^2 - m^2\Omega_A^2) D^* \left(\frac{u}{N}\right)\right]$$

$$- \left\{2mrD[\rho_o(N\Omega - m\Omega_A^2)] + \rho_o m^2 (N^2 - \Phi + 4\Omega^2 - m^2\Omega_A^2 + \Psi_A)\right\}\left(\frac{u}{N}\right) = 0 \quad (25)$$

$$\text{for } R \leqslant r < \infty.$$

Multiplying Eq. (25) by $r\left(\frac{\bar{u}}{\bar{N}}\right)$, integrating the resultant equation over the outer region, and applying the boundary condition at infinity, we obtain

$$\left[r\left(\frac{\bar{u}}{\bar{N}}\right)\rho_0 r^2 (N^2 - m^2\Omega_A^2) D^* \left(\frac{u}{N}\right)\right]_{R+o}$$

$$+ \int_R^\infty \rho_o (N^2 - m^2\Omega_A^2)\left[r^2 \left|D^*\left(\frac{u}{N}\right)\right|^2 + m^2 \left|\frac{u}{N}\right|^2\right] rdr$$

$$+ \int_R^\infty \left\{2mrND(\rho_o\Omega) - m^2[r\Omega^2(D\rho_o) + rD(\rho_o\Omega_A^2)]\right\}\left|\frac{u}{N}\right|^2 rdr = 0. \quad (26)$$

Using the complex conjugate of Eq. (3) and substituting Eqs. (22), (23), and (25) into (26), we obtain

$$a_m\left(\frac{\omega}{m}\right)^2 - 2b_m\left(\frac{\omega}{m}\right) + c_m = 0 \quad (27)$$

where

$$a_m = m\rho_1 + \int_R^\infty Q_m dr$$

$$b_m = (m-1)\rho_1\Omega_1 + \rho_2\Omega_2 + \int_R^\infty \Omega Q_m dr + \int_R^\infty D(\rho_o\Omega)|\phi_m|^2 r^2 dr$$

$$c_m = (m-2)\rho_1(\Omega_1^2 - \Omega_{A1}^2) + 2\rho_2(\Omega_2^2 - \Omega_{A2}^2) - F_c/R$$

$$+ \int_R^\infty (\Omega^2 - \Omega_A^2) Q_m dr + \int_R^\infty D[\rho_o(\Omega^2 - \Omega_A^2)]|\phi_m|^2 r^2 dr$$

$$Q_m = \rho_o(r^2|D^*\phi_m|^2 + m^2|\phi_m|^2) r \geqslant 0 \quad (28)$$

and the transformation

$$\phi_m = \frac{1}{A\,R^m}\frac{u}{N}$$

has been applied in the region $R \leqslant r < \infty$. The flow will be stable if

$$b_m^2 - a_m c_m \geqslant 0. \tag{29}$$

Since Q_m and therefore a_m are positive definite, we can immediately conclude that again the sign of F_c, representing the balanced or unbalanced force condition at the interface, determines the stabilizing or destabilizing effect on the flow. The stabilility mechanism conveyed by Eq. (29) can not be observed directly in this case. To further investigate this mechanism, we utilize the transform

$$P_m = \rho_o[r^2|D\phi_m|^2 + (m^2-1)|\phi_m|^2]r \geqslant 0 \tag{30}$$

and from Eq. (28)

$$Q_m - P_m = \rho_o D(r^2|\phi_m|^2).$$

The coefficients in Eq. (27) can therefore be rewritten as

$$\begin{aligned} a_m &= m\rho_1 + \int_R^\infty Q_m dr \\ b_m &= (m-1)\rho_1\Omega_1 + \int_R^\infty \Omega P_m dr \\ c_m &= (m-1)\rho_1(\Omega_1^2 - \Omega_{A1}^2) - (m^2-1)T/R^3 + \int(\Omega^2 - \Omega_A^2)P_m dr. \end{aligned} \tag{31}$$

Substituting the coefficients in Eq. (31) into Eq. (29), we find, after some mathematical manipulations, that stability of the flow is guaranteed if

$$x_1 + x_2 + x_3 \geqslant 0 \tag{32}$$

where

$$x_1 = -\left\{(m-1)\rho_1\int(\Omega - \Omega_1)^2 P_m dr + \delta_m\right\}$$

$$x_2 = \left[\rho_2 - \rho_1 + \int(D\rho_o)r^2|u|^2 dr\right]\left[(m-1)\rho_1\Omega_1^2 + \int\Omega^2 P_m dr\right]$$

$$x_3 = \left[m\rho_1 + \int Q dr\right]\left[(m^2-1)T/R^3 + (m-1)\rho_1\Omega_{A1}^2 + \int \Omega_A^2 Q_m dr\right] \geqslant 0$$

and

$$\delta_m = \int P_m dr \int \Omega^2 P_m dr - \left(\int \Omega P_m dr\right)^2.$$

The roles played by the flow quantities on the stability mechanism can be observed by examining Eq. (32). From the Schwarz inequality that

$$\delta_m \geqslant 0 \tag{33}$$

for all values of Ω, the terms in x_1 in Eq. (32) always destabilize the flow. Furthermore, by comparing the terms in x_1 with those in the first pair of curly brackets in Eq. (18), we can draw an analogy between the two and conclude that they both convey the shear effect which destabilizes the flow. The first term in x_1 is the shear effect generated by the velocity difference at the interface. The second term in x_1 represents the shear effect carried by the tangential velocity in the outer region and is similar to the corresponding term in Eq. (18) except that the latter is induced by the axial velocity instead. The term x_2 in Eq. (32) is the contribution to stability by the density variation in the centrifugal force field. Obviously the stabilizing or destabilizing effect depends on the density difference at the interface and on the density distribution in the outer region. Densities that increase with radius always stabilize the flow and vice versa as one would intuitively expect. The above discussion reveals that the angular velocity in rotating flows plays a dual role in flow stabilities: producing a shear effect and inducing a centrifugal force field. The term x_3 in (32) contains the information on the surface tension and on the magnetic field. The surface tension always stabilizes nonaxisymmetric perturbations as is well-known. The azimuthal magnetic fields in both the inner and outer region always stabilize the flow in spite of the details of the magnetic profile. This characteristic is also true for arbitrary flows if only the perturbations in the azimuthal direction are permitted.

To further illustrate the stability characteristics described by Eq. (32), we consider a special flow profile

$$\begin{aligned} &\rho_o(r) = \rho_2 \\ &\Omega(r) = \Omega_2 \qquad \text{for } R \leqslant r < \infty. \\ &\Omega_A(r) = \Omega_{A2} \end{aligned} \tag{34}$$

All the quantities with numerical indices are constants. The solution, obtained by solving Eq. (25) for the flow profile given in Eq. (34), is

$$u_2 = BmN_2\left[\frac{2(N_2\Omega_2 - N_{A2}\Omega_{A2})}{N_2^2 - N_{A2}^2} - 1\right]r^{-m-1}. \tag{35}$$

Equation (29) for this flow profile has the form

$$(\rho_1 + \rho_2)\omega^2 - 2[(m-1)\rho_1\Omega_1 + (m+1)\rho_2\Omega_2]\omega + m[(m-2)\rho_1(\Omega_1^2 - \Omega_{A1}^2)$$
$$+ (m+2)\rho_2(\Omega_2^2 - \Omega_{A2}^2) - F_c/R] = 0. \tag{36}$$

As previously predicted, the sign of F_c determines whether or not the force condition at the interface stabilizes the flow. For stability ($\omega_i = 0$) the characteristic equation from (36) requires that

$$-(m^2-1)\rho_1\rho_2(\Omega_1 - \Omega_2)^2 + (\rho_2 - \rho_1)[(m-1)\rho_1\Omega_1^2 + (m+1)\rho_2\Omega_2^2]$$
$$+ m(\rho_1 + \rho_2)[(m-1)\rho_1\Omega_{A1}^2 + (m+1)\rho_2\Omega_{A2}^2 + (m^2-1)T/R^3] \geqslant 0. \tag{37}$$

As previously discussed, the first term in the above equation represents the shear effect at the interface, always destablizing the flow. Because of the uniform rotation for $r \geqslant R$, Eq. (33) is identically equal to zero, meaning that no shear effect exists in the outer region. The second term in Eq. (37) is the effect of the density difference experienced in the centrifugal forces generated at the interface. Stabilizing effects correspond to larger density in the outer region. The last term in Eq. (37) contains the information on the surface tension and on the azimuthal magnetic fields in the inner and outer region, all always stabilizing azimuthal perturbations. Figures (2a,b,c) show the stability domains for $m = 2$, 5 and 30 modes in the absence of surface tension. Both the destabilizing effect produced by the shear at the interface and the stabilizing effect induced by the azimuthal magnetic fields increase as the wave number becomes larger. For very large m, Eq. (37) for zero surface tension reduces to

$$-\rho_1\rho_2(\Omega_1 - \Omega_2)^2 + (\rho_1 + \rho_2)(\rho_1\Omega_{A1}^2 + \rho_2\Omega_{A2}^2) \geqslant 0. \tag{38}$$

In the absence of the magnetic force, the flow is always unstable except for uniform rotation where $\Omega_1 = \Omega_2$.

Case 3: The arbitrary mode

For simplicity consider $\Omega_1 = \Omega_{A1} = 0$ in the inner region, and the solution for the perturbation velocity reduced from Eq. (7) is

$$u_1 = AN_1kI_m'(kr) \tag{39}$$

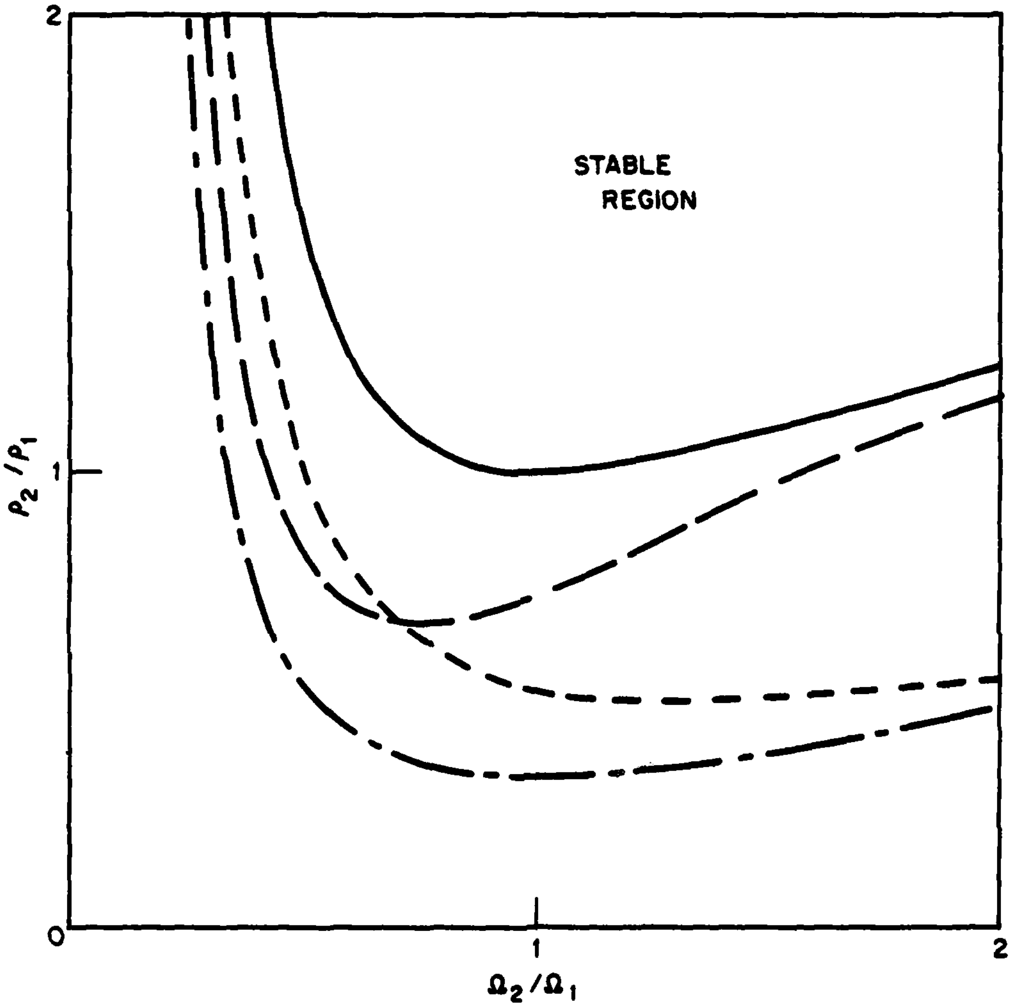

Fig. 2a. Stability domains for vortex flows subject to the $m = 2$ azimuthal perturbation (—— $\Omega_{A1} = \Omega_{A2} = 0$; — — — $\Omega_{A1}/\Omega_1 = 0.5$, $\Omega_{A2} = 0$; ---- $\Omega_{A1} = 0$, $\Omega_{A2}/\Omega_2 = 0.5$; — · — · — $\Omega_{A1}/\Omega_1 = \Omega_{A2}/\Omega_2 = 0.5$).

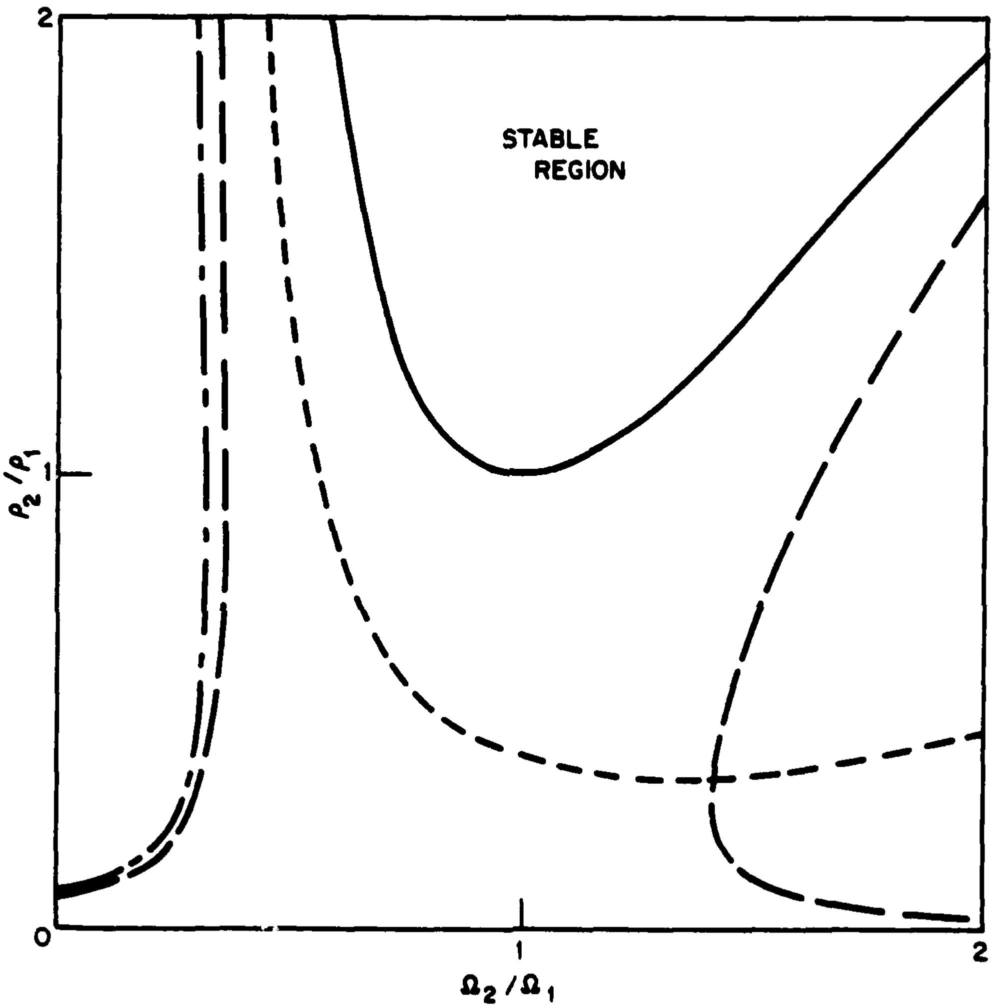

Fig. 2b. Stability domains for vortex flows subject to the $m = 5$ azimuthal perturbation (—— $\Omega_{A1} = \Omega_{A2} = 0$; – – – $\Omega_{A1}/\Omega_1 = 0.5$, $\Omega_{A2} = 0$; ---- $\Omega_{A1} = 0$, $\Omega_{A2}/\Omega_2 = 0.5$; — · — · — $\Omega_{A1}/\Omega_1 = \Omega_{A2}/\Omega_2 = 0.5$).

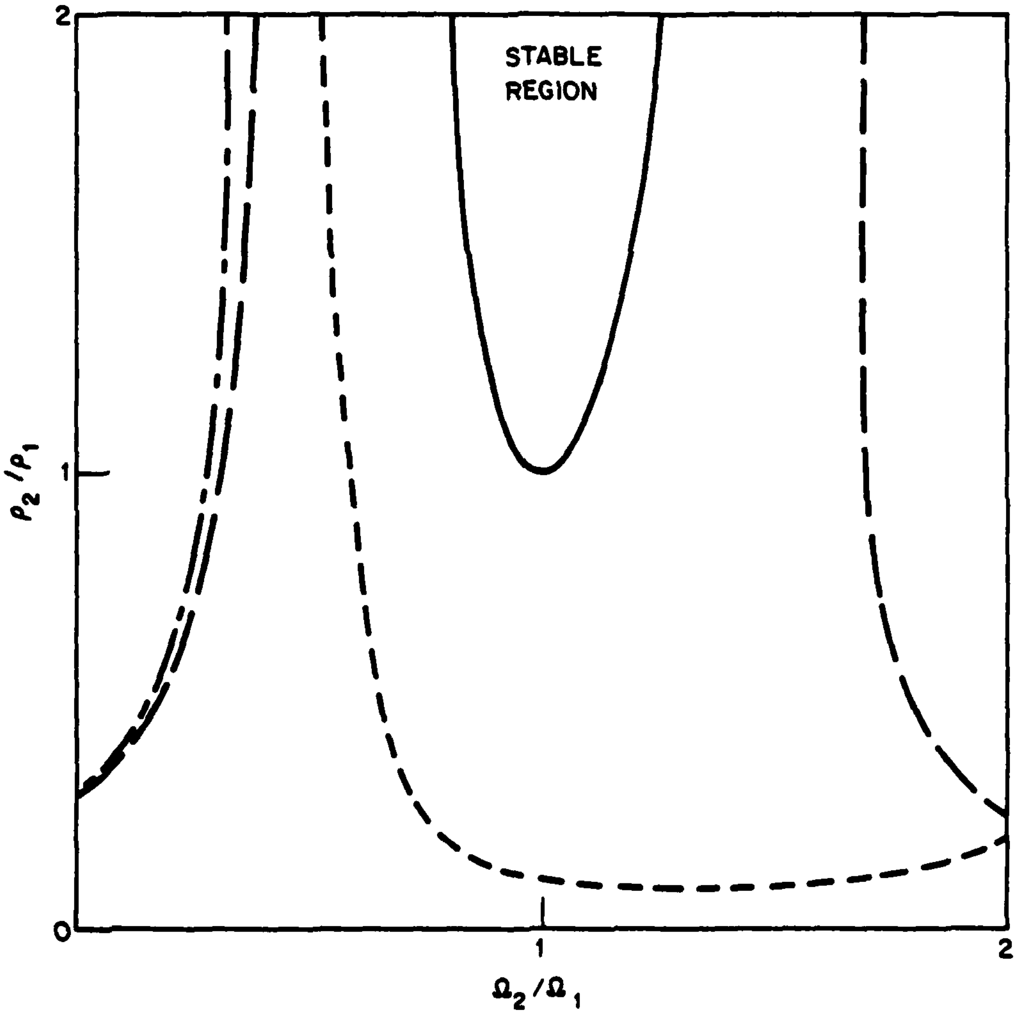

Fig. 2c. Stability domains for vortex flows subject to the $m = 30$ azimuthal perturbation (—— $\Omega_{A1} = \Omega_{A2} = 0$; — — — $\Omega_{A1}/\Omega_1 = 0.5$, $\Omega_{A2} = 0$; ---- $\Omega_{A1} = 0$, $\Omega_{A2}/\Omega_2 = 0.5$; — · — · — $\Omega_{A1}/\Omega_1 = \Omega_{A2}/\Omega_2 = 0.5$).

and

$$D^*u_1 = AN_1\left(k^2 + \frac{m^2}{r^2}\right) I_m(kr). \tag{40}$$

The dynamic interfacial condition expressed in terms of the perturbation velocities is

$$<\rho_o r^2(N^2 - N_A^2)\, D^*\left(\frac{u}{N}\right)> + \left(\frac{u}{N}\right)_R \{<-\,2mr\rho_o(N\Omega - N_A\Omega_A)> + (\kappa^2 + m^2)F_c\} = 0. \tag{41}$$

For the outer region, when we consider $\Omega_A = W_A = 0$, the governing equation for stability is

$$D\left[\frac{\rho_o r^2 N^2}{m^2 + k^2r^2}\, D^*\left(\frac{u}{N}\right)\right] - \{2mrD\left(\frac{\rho_o \Omega N}{m^2 + k^2r^2}\right) + \rho_o\left(N^2 - \Phi + \frac{4m^2\Omega^2}{m^2 + k^2r^2}\right)\}\left(\frac{u}{N}\right) = 0. \tag{42}$$

Multiply Eq. (42) by $r\left(\frac{\bar{u}}{\bar{N}}\right)$ and integrate the resultant equation over the outer region to obtain

$$\left[r\left(\frac{\bar{u}}{\bar{N}}\right)\frac{\rho_o r^2 N^2}{m^2 + k^2r^2}\, D^*\left(\frac{u}{N}\right)\right]_R^\infty - \int_R^\infty \{2mrD\left(\frac{\rho_o\Omega N}{m^2 + k^2r^2}\right)$$

$$+\rho_o\left(N^2 - \Phi + \frac{4m^2\Omega^2}{m^2 + k^2r^2}\right)\}\left|\frac{u}{N}\right|^2 rdr - \int \frac{\rho_o r^2 N^2}{m^2 + k^2r^2}\left|D^*\left(\frac{u}{N}\right)\right|^2 rdr = 0.$$

(43)

Substitute Eqs. (39), (40), and (41) into (43) and utilize the transforms

$$Q = \rho_o\left[\frac{r^2}{m^2 + k^2r^2}\,|D^*\phi|^2 + |\phi|^2\right] r \geqslant 0 \tag{44}$$

$$P = \frac{\rho_o}{m^2 + k^2r^2}\,[r^2|D\phi|^2 + (k^2r^2 + m^2 - 1)\,|\phi|^2]r \geqslant 0 \tag{45}$$

where

$$\phi = \frac{1}{A}\,\frac{u}{N}$$

has been applied in the region $R \leqslant r < \infty$. From Eqs. (44) and (45), it follows that

$$Q - P = \frac{\rho_o D(r^2|\phi|^2)}{m^2 + k^2r^2}. \tag{46}$$

The characteristic equation obtained by using the transformation from Eq. (44) to (46) is

$$a\omega^2 - 2b\omega + c = 0 \tag{47}$$

where

$$a = \alpha + \int_R^\infty Q\,dr$$

$$b = \alpha k W_1 + \int_R^\infty (kWQ + m\Omega P)\,dr$$

$$c = \alpha k^2(W_1^2 - W_{A1}^2) - [\kappa I_m(\kappa)]^2 \left[\frac{F_c}{R} - \frac{m^2\rho_2\Omega_2^2}{m^2+\kappa^2}\right] + \int_R^\infty (k^2W^2Q + 2kWm\Omega P + m^2\Omega^2 P)\,dr$$

$$- \int_R^\infty \left[\Phi + \frac{2m^2\Omega^2}{m^2+k^2r^2}\right] \frac{k^2r^2}{m^2+k^2r^2} |\phi|^2\, r\,dr$$

and

$$\alpha = \rho_1 \kappa I_m'(\kappa) I_m(\kappa) \geqslant 0.$$

The role played by the unbalanced forces arising from the discontinuities at the interface can be observed immediately. Since both α and Q are positive definite, the sign of F_c therefore determines the stability effect carried by the forces acting at the interface. In the present case

$$F_c = \rho_2 R\,\Omega_2^2 + T(\kappa^2 + m^2 - 1)/R^2.$$

The surface tension always stabilizes nonaxisymmetric perturbations as is well-known. The centrifugal force term arising from the difference in the angular velocity at the interface always stabilizes the flow. This is also the term omitted by Michalke & Timme (1968) in their stability analysis of two-dimensional vortex type flows. Such an omission led them to an erroneous conclusion that all azimuthal modes were unstable.

Solving the quadratic equation (Eq. (47)) for ω, we conclude that the flow will be stable if

$$y_1 + y_2 + y_3 + y_4 \geqslant 0 \tag{48}$$

where

$$y_1 = -\alpha \int_R^\infty k^2(W_1 - W)^2\, Q\,dr - \delta_w - \delta_\Omega$$

$$y_2 = 2\alpha \int_R^\infty k(W_1 - W)\, m\Omega P\,dr + 2\left[\int_R^\infty kWQ\,dr \int_R^\infty m\Omega P\,dr - \int_r^\infty Q\,dr \int_R^\infty kWm\Omega P\,dr\right]$$

$$y_3 = \left\{\kappa I_m'(\kappa)\,[\rho_2 \frac{\kappa I_m'(\kappa)}{m^2+\kappa^2} - \rho_1 I_m(\kappa)] + \int_R^\infty D\left[\frac{\rho_o}{m^2+k^2r^2}\right] r^2|\phi|^2 dr\right\} \int_R^\infty m^2\Omega^2 P dr$$

$$+ \left[\alpha + \int Qdr\right] \int \left[\Phi + \frac{2m^2\Omega^2}{m^2+k^2r^2}\right] \frac{k^2r^2}{m^2+k^2r^2}\,|\phi|^2 r dr$$

$$y_4 = \left[\alpha + \int Qdr\right] \left\{\alpha k^2 W_{A1}^2 + [\kappa I_m'(\kappa)]^2 \left[\frac{\kappa^2\rho_2\Omega_2^2}{m^2+\kappa^2} + \frac{T}{R\kappa^2}(\kappa^2+m^2-1)\right]\right\}$$

and

$$\delta_w = \int Qdr \int k^2W^2Qdr - \left[\int kWQdr\right]^2$$

$$\delta_\Omega = \int Pdr \int m^2\Omega^2 Pdr - \left[\int m\,\Omega\, Pdr\right]^2.$$

From the Schwarz inequality, it follows that

$$\delta_w \geqslant 0$$

and

$$\delta_\Omega \geqslant 0$$

for all values of W and Ω. Several stability characteristics can be observed from Eq. (48) as follows.

The quantity y_1 carries the shear effects in both axial and azimuthal directions, always destabilizing the flow. The first term in y_1 is the axial shear generated by the velocity difference between the inner and the outer regions. The second term δ_w and the third term δ_Ω in y_1 are respectively the shear effects produced by the axial and azimuthal velocity difference within the outer region. The corresponding terms can be found in the case for the axisymmetric mode and for the azimuthal mode.

The quantity y_2 is the shear effect interaction between the axial and azimuthal directions. The first term in y_2 is the interaction between the inner and outer regions, while the second term is the interaction within the outer region. All the terms in y_2 can be positive or negative, depending on the signs of the velocities and wave numbers, and therefore can stabilize or destabilize the flow. Such dependence implies whether or not the axial or azimuthal shear reinforces each other and whether or not the direction of perturbations strengthens the resultant shear effect.

The first term in y_3 is the effect of density variation at the interface and in the outer region in the centrifugal force field created by the rotation of the fluid. Densities increasing radially outwards

stabilize the flow. The second term in y_3 involves the integration of the Rayleigh-Synge discriminant over the outer region. The condition for centrifugally stable profile, i.e., $\Phi \geqslant 0$ is the precondition for the sufficiency condition of stability for flows subject to perturbations in both the axial and azimuthal directions (Fung & Kurzweg 1975). Positive values of Φ stabilize the flow.

The quantity y_4 carries the information on the forces acting at the interface. The presence of the surface tension and of the axial magnetic field in the inner region always stabilize the flow. The term involving $\rho_2\Omega_2^2$ in y_4 is the centrifugal force created by the rotation of the outer region at the interface and is always positive. Because of the nonrotating core considered in the inner region, any rotation immediately outside the interface will have stabilizing effects.

To further illustrate the stability characteristics described by Eq. (48), we consider the following profile

$$\left.\begin{aligned} \rho_o(r) &= \rho_2 \\ W(r) &= W_2 \\ \Omega(r) &= \Omega_2(R/r)^2 \end{aligned}\right\} \quad \text{for } R \leqslant r < \infty.$$

Here ρ_2, W_2 and Ω_2 are constant. The solution for the perturbation velocity obtained from Eq. (42) for the present flow profile is

$$u_2 = B(kW_2 + m\Omega_2 - \omega)\ k\ K_m'(kr). \tag{49}$$

Equation (48) then has the form

$$-\rho_1\rho_2[k(W_1 - W_2) - m\Omega_2]^2 + \left[\rho_2\frac{\kappa I_m'(\kappa)}{I_m(\kappa)} - \rho_1\frac{\kappa K_m'(\kappa)}{K_m(\kappa)}\right]\left[\rho_1 k^2 W_{A1}^2\ \frac{I_m(\kappa)}{\kappa I_m'(\kappa)}\right.$$
$$\left. + \rho_2\Omega_2^2 + \frac{T}{R^3}\ (\kappa^2 + m^2 - 1)\right] \geqslant 0. \tag{50}$$

The first term in Eq. (50) is the shear effect created by the velocity difference at the interface. The axial velocity difference always destabilizes the flow. However, such destabilization interacts with the radial shear effect generated by the rotation of the outer region. Whether or not the interaction reinforces the destabilization depends on the direction of the axial and azimuthal velocities and of the axial and azimuthal perturbations. Since $\kappa I_m'(\kappa)/I_m(\kappa) \geqslant 0$ and $\kappa K_m'(\kappa)/K_m(\kappa) \leqslant 0$, the second term in Eq. (50) is always positive and is contributed by the axial magnetic field in the inner region, the rotation of fluid in the outer region, and the surface tension at the interface. All these contributions stabilize the flow.

It is the contribution from the unbalanced centrifugal force $\rho_2 \Omega_2^2$ resulting from the discontinuities of the velocity and density at the interface neglected by Michalke & Timme (1967) and by Leibovich (1969) in their analyses of two types of vortex sheet flows. Such negligence led them to conclude that some of the perturbations which should have been stabilized by the unbalanced centrifugal forces at the vortex sheet were unstable.

CONCLUSION

The present analysis has demonstrated several characteristics of vortex flows with variable densities. Unlike the velocity in the two-dimensional shear flows, the rotation of vortex motions plays a dual role in flow stability: producing shear effects which always destabilize the flow and generating a centrifugal force field which can stabilize or destabilize the flow. The stabilization or destabilization depends on whether or not the force field is centrifugally stable. For flows of the vortex sheet type, the centrifugal force arising from the discontinuities in the rotating velocity and the azimuthal magnetic field at the vortex sheet has significant influence on flow stability. The resultant direction of the centrifugal force at the interface, dictated by the sign of F_c in Eq. (5), determines whether such force stabilizes or destabilizes the flow. As shown in Eq. (6), the forces acting at the vortex sheet interact with the perturbation displacement of the deformed interface. Smaller displacements correspond to larger wave numbers. For a given centrifugally stable flow profile on both sides of a fluid layer a smaller thickness produces less centrifugal stabilizing effects and at the same time generates sharper velocity gradients, both destabilizing the flow. This argument explains why a flow of the vortex sheet type is most susceptible to instability corresponding to disturbances with large wave numbers. For perturbations with smaller wave numbers, the magnitude of the displacement is greater and the resultant interaction will have more significant effects on flow characteristics. Such effects will stabilize perturbations corresponding to smaller wave numbers as shown in the examples in the present analysis.

The deformation of the vortex sheet affects the flow in two ways: disturbing the total pressure field and perturbing the centrifugal force field created by the azimuthal components of the velocity and the magnetic flux. The latter, even though it seems to be straightforward, is easily overlooked as in the studies performed by Michalke & Timme (1967) and by Leibovich (1969). Failure to consider such a perturbation to a stable centrifugal force at the vortex sheet can lead to the erroneous destabilization of certain modes corresponding to smaller axial and azimuthal wave numbers.

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